# Another Euclid

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A lecture prepared for presentation at
St. John’s College, Santa Fe
September 4, 2015

Euclid presents the reader with proofs about eyes and what they see. He gives us proofs about the height of trees and the depth of ditches. Some of the parts of the proofs are sunlight and shadows and mirrors and chariot wheels. Euclid offers comments about natural beings that move and grow. He offers comments that are phenomenological. He even offers comments about the workings of the human mind. But this Euclid of whom I am now speaking is not the writer and thinker that many of you may think you know. The Euclid of whom I am speaking is the author of the work known as the *Optics*. This Euclid can be found just a few inches farther along the bookshelf from the Euclid of the *Elements*. The tradition that passes both works down to us tells us that the author of both works is the same. Yet, the Euclid of the *Optics* does not seem to think the same things that I have heard many students of the Euclid of the *Elements* claim about that author. One of the most striking and persistent claims I have heard is that in the *Elements* Euclid is not making any claims about the "real" world. The Euclid of the *Optics* does not seem to think straight lines or other geometrical figures possessing similar degrees of perfection are not part of the "real" world. Rather this Euclid seems to think our power of perception to be inadequate to the perfections of the beings that are in principle perceptible. This Euclid does not seem to present mathematics as a bunch of made up things, rather he presents human beings as beings that make up some part of their experiences in order to compensate for deficiencies of our powers of perception in some cases and in order to avoid the unsettling consequences of purer or more honest perceiving on the other.

If we trust that what has been left to us as Euclid's work is the work of the same author, then we must find a way to understand this other Euclid as the same human being as the one with which we feel familiar. We are aided in this effort, and our trust is bolstered, by the fact that so much of the geometry we find in the *Optics* resembles in presentation and method and assumptions the geometry of the *Elements*.

## 1. Horoi

The beginning of the work should certainly give us a sense of the familiar. Just as he does in his *Elements*, Euclid begins his *Optics* with a set of *Horoi*, or boundaries, a term which usually get translated as and referred to as definitions. For his *Optics*, Euclid only states seven of these *Horoi*, but one should probably not regard these stated definitions as the only definitions of the *Optics*. The proofs in the Optics presuppose, in the background, a geometry that must be much like that found in the *Elements*. I say much like because the order of composition is unclear, and some tradition exists that regards the *Optics* as an earlier work. The *Horoi* of the *Optics* seem to augment the *Horoi* of the *Elements* by stating the geometrical character of selected aspects of sight. The seven *Horoi* read in the Greek as one long connected sentence, and I want to caution you not to be too quick to think of them as assumptions or presuppositions; they may very well show themselves in the course of this lecture to be conclusions, or even to be one complex conclusion.

These are the seven *Horoi* of Euclid's *Optics* as found in the only published translation in English, that of Harry Edward Burton in the May, 1945 edition of the *Journal of the Optical Society of America*. I have made some changes in the translation based upon my own consultation of the Greek original.

1. Let it be assumed that lines drawn straight from the eye carry through a space of great extent;

2. and that the figure of the space included within our vision is a cone, with its apex in the eye and its base at the extremities of the thing seen;

3. and that those things upon which the rays of vision fall are seen, and that those things upon which the rays of vision do not fall are not seen;

4. and that those things seen within a larger angle appear larger, and those seen within a smaller angle appear smaller, and those seen within equal angles appear to be of the same size;

5. and that things seen within the higher visual range appear higher, while those within the lower range appear lower;

6. and, similarly, that those seen within the visual range on the right appear on the right, while those within that on the left appear on the left;

7. but that things seen within more angles appear to be more clear.

I will try to offer a few thoughts on these definitions. The first definition does not come out of nowhere. It is perhaps the most familiar and least noted aspect of our vision – vision occurs for us through a distance. But Euclid says more than just that in his first definition. Euclid says our vision carries through a great greatness, a *megathon megalon*. Why does he think this? We ourselves think vision occurs through a distance because we tend to think of ourselves and the place of ourselves as bound up within the confines of our own body's extension.[1] Thus we note that vision shows us things to which we need to reach out in order to make contact with them. Or we need to move our whole body, for some time, in order to get over near the things we have seen from afar. But even this does not exhaust our sense that vision carries through distance, for we can see things that we never do move to or are never able to move to and interact with by extravisual means. We can see stars, and we can conclude from the way that they behave in our vision, as Ptolemy concluded, that they must be very far away.

In this first definition, Euclid also seems to think that our power of vision operates along straight lines. Why does he think that? Perhaps it may boil down to things like the way opaque objects can block our vision of things behind them, or the fact that we cannot see the back of our own head, that convince Euclid that the power of sight operates along straight lines. It seems clear we cannot derive our notion of straightness here from the testimony of our sense of sight without being tautological. Here, as in the *Elements,* we must already have a grasp of what straightness is in order to understand the definitions that involve straightness.

Since there is no limit in principle to how far a straight line may be extended in space, the first definition does not announce a limit built into the power of our vision as to how far we can see. Note again, we can see stars, and their distance from us, as Ptolemy has it, is so great that their remoteness functions for us as an actual infinity. But this first definition does have one more element; we see with our eyes. Straight lines striking other parts of our body do not produce vision. This constitutes a limit on our vision.

The second definition points out that our vision is limited in another way in that it does not extend in all directions at once, rather it takes on the spatial form of a cone. This may be thought of as a conical array of divergent straight lines emanating from the point of our eye. The cone may not have a mere single point as its vertex. It could have all the points of the seeing surface of the eye as the vertex and, especially at large distances, all the proofs would offer close approximations. But maybe thinking about the eye as one point is a way to think about the unity of visual perception.[2]

The third definition states that seeing only occurs within the geometrical limits of our power of sight. Only what is within the cone of our vision and is addressed by the rays of our vision is actually seen. The fourth definition defines what size is for vision. For vision, the size of a seen object is the size of the angle (or portion of the total cone) within which it is seen. With this definition "appearance" enters into the *Optics* as a theme. The word "appear" in this definition is the explicit sign of this. The fifth and sixth definitions define what ordinal location is for vision. Whether a seen object is seen as left or right, above or below is defined by the region of the cone of vision in which the object is seen. The seventh definition states that our vision of objects has variable clarity and that those objects seen within more angles are more clear. Another way to say this is that those objects addressed by more seats of the power of vision or more visual rays, assuming the rays diverge uniformly, are seen more clearly.

With these definitions understood in these ways, the reader is in a position to make diagrams representing a great variety of situations to which the seeing eye and its visual power may be subject. Moreover the reader is in a position to interpret diagrams or merely logically constructed scenarios in such a way as to determine what vision does and does not do, what it can and cannot do. So let us turn now to looking at a selection of some of the propositions that Euclid is able to carry out on the bases of the *Horoi* with which he begins.

## 2. Phenomena

Many of Euclid's proofs in the *Optics* might be said to fall into the category of "saving the appearances." That is, many of the proofs show how the geometry of what is out there in the world can give rise to the appearances that constitute the geometry of visual appearance. Two worlds are at play, and often in conflict, in the proofs. There is the world of what is and the world of what is seen. These worlds are never the same.

In the very first proposition of the book, Euclid sets out to prove that "Nothing that is seen is seen at once in its entirety." This is labeled proposition 1 on your handout. The diagram on your page represents a single eye (many of the proofs limit themselves to what is seen by a single eye) at the vertex of many straight lines representing the power of sight. In this diagram I have labeled the eye with the full name "eye;" in every subsequent diagram I have simply labeled eyes with the letter E. The straight line AG represents some side or dimension of an object being seen. The basis of Euclid's proof is very easy to see from the diagram itself. There are parts of the line AG that are not touched by the lines representing the power of sight, all the parts between A and B, B and C, C and D, D and E, E and F, and F and G. Now according to the third definition nothing that is not addressed by the power of sight is seen, so none of those parts between the lines are seen. Thus, at any one time, *hama*, some parts of the object will not be seen. For vision, there is no reading between the lines. That is really the entirety of Euclid's proof, and one can readily see from the diagram itself that Euclid's claim must be so. But what makes Euclid sure that there are gaps between the lines of sight? It seems to be a necessary conclusion from the starting point of the first definition. If our power of vision operates along straight lines, and if that power is comprised of more than one straight line, then the straight lines, starting from the eye and being distinct from one another, must diverge. And if the straight lines diverge, then at any distance from the eye itself there must be gaps between the lines of sight that leave regions that are unaddressed by the power of sight. The straightness of our power of vision necessitates that at any one time our seeing of an object must be partial.

After proving this, Euclid adds one sentence to his account that is not properly part of the proof at all. He says, "But it seems to be seen all at once because the rays of vision shift rapidly." This little comment tacked on at the end may be the most interesting sentence of the entire work. This sentence raises the specter of a kind of experience humans seem, *dokei,* to have that is neither quite the seeming, *phainesthai*, of what our power of sight gives us nor the underlying truth of the pure geometry. Euclid claims that some kind of rapid sweeping or flickering motion of the eyes, carrying with them the straight lines of vision's power, makes the power of vision address enough[3] of the parts of an object that it "seems" to us to be seen in its entirety all at once. Now what kind of phenomenon is this seeming that gives us the sense that we see whole objects all at once? Or, which is genuinely phenomenal: the logically necessary phenomenon,[4] "saved" in the proof, that results in the conclusion that we do not see any object in its entirety all at once or the phenomenon[5] that seems to appear to us because of the rapid shifting of the rays of vision? I will return later to this perplexing and pregnant one-sentence note of Euclid's. I just wish to call attention to it here, because it seems to light up the way in which the givens or starting points of this whole investigation raise questions about where we really start as human beings and what we really experience.

Some of these questions are carried over into the second proposition of the book. The diagram is on your handout labeled proposition 2. You can see from the parentheses that the very same diagram can (and will) be used for proposition 5.[6] Proposition 2 claims, "Objects located nearby are seen more clearly than objects of equal size located at a distance." Look at the diagram. Objects AB and CD are equal in size (in truth) but they are unequally distant from the eye E. Therefore AB, being closer to E, shall be seen more clearly than CD. The proof of this is, again, very evident from the diagram itself. For it is clear that angle CED is smaller than, and contained by, angle AEB. Thus object AB is seen by more angles than object CD. And according to definition 7, things seen by more angles are seen more clearly. Another and perhaps clearer way to say this is that object CD is addressed by the power of all the lines of sight contained within the angle CED, but object AB is addressed by the power of all those lines plus the power of those extra lines of sight contained within angle AEB but outside of angle CED. Thus AB, being nearer to the eye, is seen more clearly. This proof clarifies something about what Euclid says in his little note at the end of proposition 1. For although the rapid shifting of the lines of sight may make it seem to human beings that they see whole objects in their entirety all at once, even with that shifting there must still be gaps in what is addressed by the power of vision. For the whole explanation that Euclid gives in proposition 2 about why equal things seen at unequal distances are seen with different degrees of clarity presupposes gaps or unequal subjection to the power of sight. If the rapid sweeping of the lines of sight truly gave us all the parts of an object, we would see all objects with the same degree of clarity regardless of distance. Although this proposition comes second in the work, it may be the variable clarity of images that we experience as we see things in the world is a more fundamental and decisive kind of partialness than that Euclid presents to us in proposition one. Our experience of differing clarities in our vision may have given Euclid a firmer phenomenological reason to adopt the opinion that our power of vision is constituted by many separate and divergent lines of sight rather than one line rapidly and continuously sweeping over all that we see.

Such a single line of visual power sweeping continuously over objects and leaving no unaddressed gaps would also make the next proposition impossible. Proposition 3 claims, "Every object has a certain limit of distance, and when this is reached it is seen no longer." The proof of this claim can also be seen easily from the diagram. If line EA and line EB are divergent lines of sight between which is no other line of visual power, then the object AB is located at the very limit of its visibility when its extremities just lie upon lines EA and EB. An object of equal size, such as C, and situated farther away will fall between the lines of visual power and thus will not be seen at all. Thus anything of any size has a limit past which distance it cannot be seen. This limit is determined by the relationship between the size of the object and the angular divergence of the straight lines that carry the power of sight. The size of Stars must be very big and the divergence of the lines of sight must be very small in order for us to see them at such great distances. Still the size of the gaps in our power of sight is significant even at terrestrial distances.

This proposition seems also implicitly to be making the claim that an object that is addressed by only one line of sight at a time will not be seen. This may be because one line of sight may not give us enough to be noticeable. But that really begs the question about exactly how much is enough to be noticeable. Also the proposition seems to treat being touched by two lines of sight as enough for sight. But why should two lines give enough when one does not? This question becomes ever more pressing the more one takes Euclid literally to be claiming that the power of sight travels along straight lines. What can travel along a line, when a line is understood to be a breadthless length? The only meaningful answer[7] seems to be points, but can we compose the scope and extent of the images we think we see out of points that, being partless themselves, would seem incapable of conveying the parts of images? We might be tempted to think that if we are going to be capable of seeing anything we are going to have to be less literal in our understanding of Euclid's words. Is it possible that he understands the lines of sight to have some breadth? I have provided an image of one such "fat line" or "breadthed length" on your handout. That image does not correspond to anything explicit in Euclid's text. As you can see from the diagram, such fat lines are, of course, rectangles. If the power of vision does operate through many long, thin rectangles, it would operate in many ways as if it operated along straight lines. However, proposition 3 would not be strictly true. For objects smaller in size than the fat lines there would be no disappearing point. The whole of one of their sides would be addressed by a single line of vision that would continue to address the object regardless of its distance. The clarity of such a small object for vision based on fat lines would also not vary according to its distance, since only one line would address the object at any distance. Thus proposition 2 would not hold. Also proposition 1 would collapse, for the entirety of such a small object would be seen all at once by one fat line of sight. This thought reveals that the "entirety" spoken of in proposition 1 is only the entirety of one side of the object. The question of the other sides of the object is addressed in numerous later proofs dealing with the powers of two eyes together and depends upon the shape and size of the object and the location of the eyes and the distance between the eyes (See propositions 23 through 33). However, it might be that in such an imagined case as that of the fat lines I would have to say that all sides (or surfaces) of the object would be seen all at once since in enveloping the small object the fat line touchs all of its sides. And what is touched by the power of sight is seen.

The next proposition I want to show you also would not work for such fat lines. In proposition 5, Euclid claims, "Objects of equal size unequally distant appear unequal and the one lying nearer to the eye always appears larger." As I mentioned earlier, for this proposition you can use the diagram for proposition 2. Now according to definition 4, things seen within a larger angle will appear larger. Thus of two equal objects unequally distant from the eye E, the nearer object AB will appear larger because it is seen within the larger angle AEB.

Proposition 5 also means that for vision the size of an object will change when the eye changes its distance from the object. Proposition 3 also means that for vision objects cease to exist at certain distances. Proposition 2 also means that for vision objects become more or less distinct in their form as an eye changes its distance from them. Proposition 1 also means that without some modification, some flickering or shifting of the eye, and some power of retaining the images of many moments and of putting them together, vision does not give us experience of wholes.[8] Thus the phenomena of sight as Euclid presents them to the reader in the opening propositions seem to give us something unlike the world we usually assume we inhabit. If we focus just on what the power of sight gives us, then the world we see is not made up of things that are natural wholes. If we allow some rapid shifting to obscure that for us, then we still encounter things of sight that change their own clarity or distinctness of form because our eyes change location. The things also pass out of visual existence and come into visual existence because of the location and power of our eyes. Things also change their size when we move toward or away from them or they move toward or away from us. For vision it could even be said that objects do not have a size of their own apart from their relation to the location of our eyes.

Objects are also seen to have different shapes depending upon the location of our eyes. Here are some examples. Look at the ceiling. Notice how parts of the ceiling that are farther away from you occupy a lower portion of your visual cone than those parts nearer to being directly over your heads. Thus the ceiling appears to slope downward. This is shown in proposition 14. Similarly the floor slopes upward. This is found in proposition 13. On the basis of the same principles, the parts of the walls that are farther away appear to be closing in. You would notice this especially in a long hallway. This is shown in proposition 6 which is merely an application of what we just proved in proposition 5. This is what the room looks like. We usually do not notice it. But when we apply our attention we do see this.

Here are some more examples. Proposition 9 shows that rectangular objects seen from a distance will appear to have rounded corners. This is because their sharp corners disappear into the gaps in our vision. Proposition 22 shows that curved lines will appear straight when viewed by an eye that is in the same plane as the curved lines. Proposition 36 shows that at some distances and some angular orientations the wheels of chariots will not appear circular. This proposition seem to relate to how Euclid thought about ellipses. Proposition 58 shows how under some circumstances squares do not look like squares.

Thus the world as it seems to be given to us by our sense of sight is filled with things that seem to have few stable properties of their own independent of us. In all these proofs, Euclid presents the changeable, unstable world given to us by sight as merely apparent. However, the true, underlying world of pure geometry where things have their own size and shape and granular distinctness and tend to keep them regardless of how or where we move our eyes seems to become a much more prominent theme in later propositions.

## 3. The Underlying Truth

The difference between these two worlds is a very prominent theme in this next proof I wish to show you. This proof will likely be the hardest to follow of all I will present this evening, because it relies on a greater number of assumptions of previous proofs not in the *Optics* and because it involves some manipulation of ratios. Proposition 8 claims, “Lines of equal length and parallel, if placed at unequal distances from the eye, are not seen in proportion to the distances.” This proposition could also be treated as addressing the proportional change in apparent size of one object that changes its distance from the eye. Here the two worlds come starkly apart because they do not share a ratio. That is, they do not share a logos, they are *ouk analogos*.[9] This happens in part because size for the world of vision is dependent upon angles and size for the world of pure geometry is measured linearly. Now let's take a stab at proving proposition 8. You may be encouraged by the fact that many of you have seen this proof (likely without knowing it) in the work of Ptolemy. Both a part of the diagram of this proof and the heart of its reasoning is used in what is called Lemma 1.2 of the Green Lion Edition of the *Almagest* (starting at the bottom of p. 43 (which is also Heiberg's p.43)). The proof really is not that bad, just less easy than the others.

If you look at the diagram what we need to prove is that angle CEA does not have to angle DEF the same ratio as line FE has to line AE. Angle CEA is the visual size of object CA and angle DEF is the visual size of object DF. If visual size were simply dependent upon distance from the eye then angle CEA would have to angle DEF the same ratio as line FE has to line AE. However if you draw in the circular arc GBH through point B with center E, then we can see the following. Triangle EBC is bigger than sector EBH because the triangle contains the sector. A sector is a portion of a circle bounded by an angle from the center. And triangle EAB is smaller than sector EGB because there the sector contains the triangle. Thus triangle EBC has to sector EBH a larger ratio than triangle EAB has to sector EGB. Taking the alternate ratio, which Euclid proves we can do in his *Elements*, triangle EBC has to triangle EAB a greater ratio than sector EBH has to sector EGB. You will see this if you remember triangle EBC is bigger than sector EBH and triangle EAB is smaller than sector EGB, thus the ratio of the triangle that is bigger than its respective sector to the triangle that is smaller than its respective sector is greater than the ratio the sectors have to each other. Now the ratio of sector EBH to sector EGB is the same as the ratios of the angle CEB to angle DEF because sectors of a circle are proportional to the angles at the center of the circle that contains them. Now we need to manipulate one more ratio. It is a consequence of what we have already shown that the combined triangle EAC (made up from triangle EBC and triangle EAB) has to its part, triangle EAB, a greater ratio than the combined sector EGH has to its part, sector EGB. Now triangle EAC has to triangle EAB the same ratio as the ratio of their corresponding sides line AC and line AB have to each other. And since line AC is equal in length to line FD, triangle EAC has to triangle EAB the same ratio as line FD has to line AB. And line FD has to line AB the same ratio as line FE has to line AE since both pairs of lines are, respectively, the sides of similar triangles. FE is the distance from the eye to object FD and AE is the distance from the eye to object AC. And this ratio of distance FE to distance AE is, being proven the same as the ratio of triangle EAC to triangle EAB, necessarily greater than the ratio of the sector EGH to the sector EGB. But those sectors had the same ratio as the angles CEA and DEF, and those angles are the sizes of the objects for vision. Thus the apparent sizes of the objects for vision are not in proportion to the true geometric distances of the objects from the eye. I hope you were able to follow that. Regardless of how clear I have been able to make the proof of this proposition for you, I want you to see a consequence of this proof. Size has a different and somewhat independent meaning in the visual world of appearance than it has in the true world of pure geometry. Size in the visual world depends upon angular measure. Now of course angles and their measure have their own meaning in the world of pure geometry, but that meaning does not determine the size of things other than that of the angles themselves.

Despite this separation of the logoi of the two worlds, Euclid does present a series of proofs that show how the appearances of the visual world can be used to calculate the true size of unknown (but seen) objects. You will see these on your handout in the trio of diagrams labeled propositions 18, 19, and 21. All of these propositions operate by coming to know the true size, somehow, perhaps by measurement with some standard physical measuring device, of three lengths and then finding a fourth proportional. In proposition 18, if you want to know the height of an unknown object DC when the sun is shining and casting a shadow of that object then all you need to do is situate your eye on the ground at the extremity of the shadow and interpose a known object AB, oriented perpendicular to the shadow on the ground, into the shadow until it is placed at that point just where the height of the known object is wholly, but just, within the shadow. Since you already know the length of AB and you can come to know (by measuring) the distance EA from the eye to AB and the distance EC from the eye to DC, and since EA has to AB the same ratio as EC has to the unknown DC, all you need to do is calculate the fourth proportional. Proposition 19 enables one to find an unknown height when the sun is not shining in such a manner as to cast a useable shadow. There a standing man identifies that point in a mirror placed at ground level at which the upper extremity of the unknown object DC appears. Because, as Euclid knows, the angle of incidence equals the angle of reflection at the mirror, similar triangles are formed by the standing man and unknown object and the mirror image. Once again one need only measure three lengths, the height of the man’s eye, the distance from the man’s foot to the top of the image in the mirror, and the distance from the top of the image in the mirror to the base of the unknown object. After that, one need only calculate the fourth proportional to find the height of the unknown object. Proposition 21 may even be simpler. In order to find the length of unknown object CD one merely needs to move a known object AB into the angle in which CD is seen until the extremities of AB just exactly block your sight of the extremities of CD. Then one merely needs to find the measure of distances EA and EC and one is again in a position to find the fourth proportional. Thus by means of the appearances of the visual world, Euclid shows he can calculate the true geometric size of an unknown (but seen) object.

These proofs are followed by another set of proofs that show vision to be partial or incomplete in a manner different from the partiality presented in proposition 1. These proofs also continue to show a divide between the visual world and the true world, for these proofs all maintain that there is more to a seen object than those portions or sides that are or can be addressed by the power of vision. This, too, raises a question about whence we acquire our notions of wholeness, and also a question about what kind of notions those are.

Proposition 23 considers what happens when an eye looks at a sphere. The sphere is a good place to start because it would seem on the basis of its shape to allow an eye to see more of it at one time than every other shape. The sharp corners of solids bounded by flat surfaces tend to block the vision of eyes that see along straight lines. Despite the seeming promise offered by the shape of the sphere, proposition 23 claims, “Of a sphere seen in whatever way by one eye, less than a hemisphere is always seen. . .” If you glance at the two dimensional diagram on your handout you can see why this is so. The straight lines emanating from the eye cannot address any part of the surface of the sphere beyond the points where those straight lines become tangent to the sphere. And those points of tangency always occur before the power of sight can address so much as a hemisphere, for at the hemisphere the two tangents would both be perpendicular to the same diameter. That would make triangle ABE a triangle whose angles added up to more than two right angles, which cannot be (for Euclid). Now propositions 25 through 27 show what you can see when you look at a sphere with two eyes. I have only given you the diagram for proposition 25 on your handout. In these propositions everything depends upon the size of the sphere compared to the distance between the two eyes. In proposition 25 the distance between the eyes is the same as the diameter of the sphere, and in that case you can see exactly the hemisphere. Proposition 26 considers cases where the distance between the eyes is greater than the diameter of the sphere. In such cases, for which you can imagine your own diagram, you can see more than the hemisphere, but never quite all of the sphere. Proposition 27 shows that if the distance between the eyes is less than the diameter of the sphere then you can only see less than a hemisphere. Thus if the visual world and its possibilities is based upon a being whose power of vision is limited to only two eyes, the objects of the true geometric world will always be keeping something hidden from the power of vision.

Next I want to show you a proposition the enunciation of which begins like a fairy tale and the content of which behaves like a kind of magical charm. Euclid begins proposition 38 *esti tis topos*, “there is some place.” It is like a children’s tale about a special region in space that has its own special properties. “There is some place where, if the position of the eye is changed, while the thing seen remains in the same place, the thing seen always appears of the same size.” Unlike what happens elsewhere, when the eye moves around within this special place the visual size of the object does not change. You have the diagram for proposition 38 on your handout. If AB is the seen object, then AB looks the same size as seen from E1 or E2 or E3 or from anywhere else along the curve. This is because the curve is the arc of a circle cut off by the chord AB. Since all the angles in a segment of a circle are the same in measure, an eye placed along the arc of the circumference and looking at the chord that cuts off the circumference will always see the chord within the same angle. And the same angle means the same size. While the linear distance from the eye to the chord will vary along the arc, the visual size of the chord will not vary. This means that visual size does not depend upon linear distance. This proof does not seem to establish that objects viewed from such special places will have parts that all appear with the same degree of clarity however. Visual clarity seems to depend not so much on how many rays of vision touch an objects as on how large the gaps are between the rays. In this way the visual clarity of an object does seem to depend upon distance alone, but visual size is determined by the factors that determine angular size, and angular size is certainly not determined by distance alone.

Euclid makes a very clever extension of what is shown in proposition 38 to show in proposition 45 that there is a place where the eye can be located to make two objects of different true sizes appear to have the same size to the eye. The diagram for proposition 45 is on your handout. If two objects of unequal sizes are arranged in a straight line there is a place from which the two objects will look equal to the eye. Euclid finds this special place by a simple extension of the principle shown in proposition 38. If one constructs on each of the two objects similar arcs cut off by the objects as chords, with the stipulation that the arcs are greater than semi-circles (this ensures the two arcs will intersect), an eye placed at the intersection of the two arcs will see the unequal objects to have the same size. This is because the angles in similar segments are equal. Once again, equal angles mean the same visual size. And once again Euclid continues the theme that what vision gives us is not the truth, although the givens of vision can be accounted for by the underlying geometric truths.

In proposition 51 Euclid throws in a proof about relative motion, perhaps to show us that not only does vision deceive us about size and shape, but also about locomotion. The scenario for this proof is that three objects A, B, and C are all moving in the same direction but at different speeds. The eye is also moving in the same direction and with the same speed as object B. Object A is moving more slowly and object C is moving more swiftly. Euclid claims that the object “moving with the same speed as the eye, will seem to stand still, others, moving more slowly, will seem to move in the opposite direction, and others, moving more quickly, will seem to move ahead.” The diagram for this proposition may not seem to be very helpful, but if you use your imagination to imbue the diagram with change over time, Euclid’s proof will readily appear. With the objects and the eye moving according to their assigned speeds, the eye and object B, moving faster, will leave A behind and angle AEB will widen. A will thus appear in a part of the eye’s visual range that is more left than the part where A started. Being seen in the more left portion of the visual range will look as if motion leftward has taken place. This is a consequence of definition 6. Similarly object B will seem to stay in the same place within the visual range and object C will be seen as moving rightward.[10] Thus the appearances given to us by vision will deceive about both what is or is not in motion and about the way things are in motion.

Visual appearance will even deceive us about alteration, as Euclid shows in propositions 53 and 56. Proposition 53 claims that, “When the eye moves nearer the object seen, the object will seem to grow larger.” The proof of this claim can be seen easily from the diagram on your handout. In moving from E1 to E2 it is clear that the eye comes to see object AB in a larger angle at its second position. If one imagines the whole movement of the eye to be a continuous approach nearer to the object, then the diagram represents the continuous motion analyzed into each moment of the motion. Thus the visual appearance would be one of a continuous getting larger of the object. Such growth or enlargement is the kind of *kinesis* classically categorized as alteration. Of course this object is not altering; it just seems so to the moving eye. Proposition 56 compliments the scenario of proposition 53 by claiming, “Objects increased in size will seem to approach the eye.” The actual proof that Euclid offers of this is quite odd. Basically, he says that when the object grows it will occupy a greater angle of vision. This is fine and geometrically sound. But after that, Euclid seems to offer psychological observations based upon some kind of assumption that we get used to things appearing to get bigger within our cone of vision as they get closer to us. And we acquire thereby a habit of thought that comes to conclude if an object gets bigger in our sight it is coming toward us (but if we come to think this way we will not be able to distinguish between alteration and locomotion). His actual words are, "But things thought to be greater than themselves seem to be increased, and the things nearer the eye appear greater.[11] So objects increased in size will seem to approach the eye." The word for "appear" in this passage is phainetai and "seem" in each case is some version of dokei. The two species of phenomena found at the end of proposition 1 are showing themselves again. Perhaps then this puts us in a position to return to the promised discussion of Euclid's one-sentence note to proposition 1.

## 4. Commitments

One might have thought that whatever world we inhabit would have to be given to us by what our senses give us. But throughout the *Optics* Euclid does not treat what sight gives us as a foundation or a starting point. It should be made clear that Euclid does not appear in this text to be correcting sight here by means of the testimony of some other sense such as hearing or touch. What then allows Euclid to push aside the testimony of the sense of sight in favor of a world that differs in its character in many ways?

We might approach a beginning of an answer to this question by returning our attention to the one-sentence comment Euclid appended to his proof of proposition 1, "But it seems to be seen all at once because the rays of vision shift rapidly." A lot seems to be contained in this brief comment. In order for some rapid shifting of the rays of vision to seem to us to overcome the partial character of our power of vision a number of things seem to be necessary. We must have other powers. We must have a power of holding onto moments of vision (it is tempting to call this memory). We must also have a power that allows us to put together those moments into a seeming whole (it is tempting to call this imagination). Somehow a series of moments in time must be put together to give the appearance of one human moment of time. And a series of moments of vision that seem to have presented themselves to the eye sequentially must be held onto and put together to give the appearance of the experience of the sight of a visual whole. Euclid leaves a lot here for us to guess at, and there is no way to be sure about the details. We can, however, draw reasonable conclusions about the sort of things that must underlie the thinking present in his one-sentence comment.

There may be no avoiding the conclusion that Euclid thinks the wholeness of things that seems to us to be part of our experience of sight is really some product of our own making. That is not to say that the wholeness is something wholly made up and does not correspond to any real thing. But it is to say that for Euclid our experience of wholes is not given to us by our sense of sight (or any other sense). Our experience of wholes is given to us by something we do to what is given us by our senses. Something like memory must hold onto the moments of vision. Something like an image making capacity must synthesize those moments together so that the mind addresses them as one thing. This holding together and putting together must operate according to rules that determine what to put together, what to keep separate, when to start, when to stop. These rules for wholeness cannot be given to us by our partial power of sight, which in its very partiality does not give us wholes. If I were to call these rules schemas, one might be tempted to suppose that what Euclid has in mind is something like the transcendental deduction found in Kant's *Critique of Pure Reason*, especially in the threefold synthesis of apperception in the first edition. In one little sentence, Euclid prefigures Kant. How far might that thought go? Is Euclid already thinking that the wholeness that seems to be available to us in our experience of seeing things (and that corresponds to the wholeness Euclid ascribes to objects in his underlying true geometry) is really to be derived from a priori sources such as a formal intuition of space and inborn categories of logical reasoning?

The a posteriori option of deriving the underlying true geometry from what sight gives us by means of some abstraction would seem to be ruled out by the following consideration. The term "abstraction" means most originally something that has been "cut away" from something else. By means of such cutting away a thing is made lesser, not more. But the experience of wholeness that Euclid presents as seeming to be available to our human awareness of sight is fuller or more than the partialness that his first proposition concludes belongs to our power of sight alone. Thus abstraction, which makes things ever lesser, cannot be the way to get to the fuller objects that populate Euclid's supposed world of pure geometry.

Now in his transcendental deduction Kant describes a series of happenings or events that occur below the threshold of human experience or awareness. Perception gives us something, a manifold, but something must be done to that given in order to make it capable of being experienced. The manifold given us by perception must be made more thought-like or more concept-like before we can be aware of it, before we can think it. Euclid's little sentence at the end of the first proposition seems to indicate our experience of sight, the way sight seems to us, involves a wholeness that is more than what the power of sight itself can logically be concluded to give us. Euclid also seems to think that such wholeness belongs properly to the world underlying the world revealed by our senses. It may be that our thinking experience, our awareness, is necessarily shaped by the inborn givens of some kind of a priori nature, but why commit one's thought to the additional conclusion that the true world must correspond to our way of thinking as well?

For Euclid the world of visible objects seems to be something like see-able space. All the see-ables are continuous magnitudes. Continuity means there is potentially infinite divisibility and a power of sight that operates along straight lines makes many divisions, but if its job is to see all that is there, that job can never be finished. If we think back to proposition 3 it seems we need more than one ray of vision to touch an object in order to have some sight of it. Seeing more than one point then, seems to engage our process for making up a whole. It may be that this process should be thought of as filling in, for we seem to make our experience of wholes by filling in the gaps not given to us by vision. This filling in process must in this account be guided by or disciplined by whatever points we do see. The more points the more guidance. But even with many points given us by vision, what we make up to fill in the gaps is never as "clear" as what is "given" us by sight. This may mean something for what we think about perceived reality, and perhaps for what we think about madness.

In presenting the pure geometry of his proofs as the underlying truth beneath the appearances of vision, Euclid seems to commit to more than Kant does. Euclid seems to be claiming to know how things are in themselves. One might say that Euclid's real geometrical beings are not Kant's inaccessible things in themselves, but they are never wholly accessed.

Why might Euclid make such a claim? It is not intrinsically crazy or inconsistent to say that things in the world change their sizes and shapes, that they sometimes become less distinct in their forms, that they sometimes pass out of existence. But most people think it is crazy to think that simply *because* one person's eye changes location all the other things in the world would suffer changes.  That is to say, I would venture a guess that it is because of a concern with causality that Euclid rejects the testimony of the power of sight and concludes that the things in the world must be more stable and independent of one another, just as they are in his pure geometry.  Euclid's interest in causality in the *Optics* is evident in all the proofs about relative motion wherein he attributes the appearances to the motion of the viewing eye, but does not present the movement of the eye as causing any real change in anything other than itself.

Now it seems to me that Euclid rejects, as most people would, the kind of causality where things at a (spooky) distance change because some unrelated thing moved or acted.  This may seem like good sound sense, but it is worth pointing out that the kind of causality being rejected is the kind of causality exercised by a deity in schemes of particular providence.[12]  This kind of causality is also found in certain interpretations of quantum mechanics.  This is also similar to the way in which Heidegger indicates that certain modes of being only show themselves to human beings who are in the proper moods (Befindlichkeiten).  I mention these examples to show that it is not impossible to think the world is more like what is given to us by sight than it is like the stable, independent, true geometry that Euclid seems to prefer.  And causality does not have to be thought of as something that happens only between objects that touch one another.

What we have looked at here in the *Optics* then would seem to raise questions such as whether Euclid is ontologically committed to a certain kind of causality and with it certain kinds of beings and certain modes of Being.  Is Euclid committed to a certain kind of scientific outlook on the world, one that carries with it a certain kind of natural theology, one that would only allow certain kinds of deities?  These kinds of questions may not come up frequently or urgently in most encounters with Euclid's *Elements*, but they are absolutely provoked by the repeated distinction between appearance and reality in the *Optics*. It may thus prove to be the case that studying the *Optics* reveals much to us about Euclid's commitments and about the grounds of those commitments. It may also prove to be the case that the *Optics* can help us learn to read Euclid's proofs better. By better I mean in such a way as to investigate these underlying things that Euclid thinks about deeply but is not trying to prove to us in his texts.

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[1] If we thought of our vision itself as our extension it would be otherwise.

[2] How many "eyes" do we have?

[3] I say "enough of the parts of an object" out of deference to the way that Euclid treats our "seeing" objects more and less clearly as a species of seeing more or less of the parts of an object at the same time in Proposition 2.

[4] By this first "phenomenon" I mean the argument of the proof that claims because our sight operates along diverging straight lines there are gaps between them where we do not see.

[5] By this second "phenomenon" I mean our impression that we do have the experience of seeing whole objects all at once.

[6] It could also be used for the second alternative proof of proposition 54.

[7] Other straight lines could travel along straight lights but their function for vision would be no different than that of travelling points.

[8] One might wonder how this affects the beginning of the *Metaphysics*.

[9] Whereas an angle is in its essence a relationship, linear size, although it can be determined in some cases by relations, is essential an independent thing.

[10] I believe what I have just said is a sound account of the reasoning of proposition 51. However, in the very next proposition, for which the same diagram may be used, it is difficult not to think that Euclid is reasoning differently. In proposition 52, Euclid claims to show that if A and C move (in the same direction) and B does not move, then B will seem to move backwards. But under these conditions B will not be any more leftward in the visual field as a result of A and C moving. Euclid seems to make the seemingly backward movement of B depend here on angle AEB getting smaller and angle BEC getting larger. This whole proof may depend upon thinking that there are no other reference points but the three seen points A, B, and C. But that would have to mean that the eye itself is not a reference point. In my interpretation of proposition 51 I am taking the eye as a reference point. I understand definitions 5 and 6 to justify this.

[11] In both manuscript A and manuscript B there seems to be an error. When both I and Burton translate "greater," following the logic, the Greek has elatttona, "lesser."

[12] There may also be implicit in this a rejection of so-called "spontaneous" causality such as serves as the basis of Kantian freedom.